the CIRCLES AND THEOREMS

Tanzania syllabus

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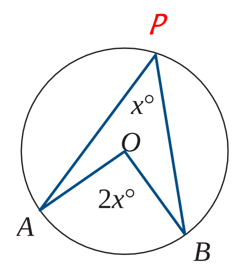
At the end of this topic, learners should be able to: -

- To establish the following results and use them to prove further properties andsolve problems:
- The angle subtended at the circumference is half the angle at the centre subtended by the same arc
- Angles in the same segment of a circle are equal
- A tangent to a circle is perpendicular to the radius drawn from the point of contact
- The two tangents drawn from an external point to a circle are the same length
- The angle between a tangent and a chord drawn from the point of contact isequal to any angle in the alternate segment
- A quadrilateral is cyclic (that is, the four vertices lie on a circle) if and only if the sum of each pair of opposite angles is two right angles
- If AB and CD are two chords of a circle which cut at a point P (which may beinside or outside a circle) then $PA \cdot PB = PC \cdot PD$
- If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB is a secant then $PT = PA \cdot PB$

These theorems and related results can be investigated through a geometry package such as Cabri Geometry.
It is assumed in this chapter that the student is familiar with basic properties of parallel linesand triangles

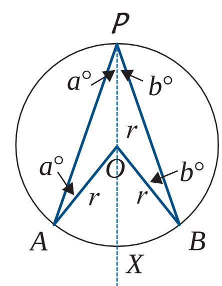
(a) ANGLE PROPERTIES OF THE CIRCLE



Theorem 1

The angle at the centre of a circle is twice the angle atthe circumference subtended by the same arc.

Proof



Join points P and O and extend the line through O as shownin the diagram.

Note that $AO = BO = PO = r$ the radius of the circle.
Therefore, triangles PAO and PBO are isosceles.

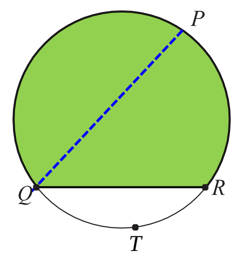
Let $\angle APO = \angle PAO = a$ and $\angle BPO = \angle PBO = b$
Then angle AOX is $2a$ (exterior angle of a triangle) and angle BOX is $2b$ (exterior angle of a triangle)

$\therefore \quad \angle AOB = 2a + 2b = 2(a + b) = 2\angle APB$

Note: In the proof presented above, the centre and point P are considered to be on the same sideof chord AB.

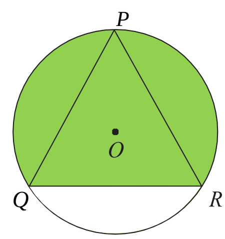
The proof is not dependent on this and the result always holds. The converse of this result also holds:

i.e., if A and B are points on a circle with centre O and angle APB is equal to half angle AOB, then P lies on the circle.



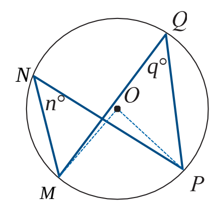
- A **segment** of a circle is the part of the plane bounded by an arc and its chord.
- Arc QPR and chord QR define a major segment which is shaded.
- Arc QTR and chord QR define a minor segment which is not shaded.

$\angle QPR$ is said to be an angle in segment QPR.



Theorem 2

Angles in the same segment of a circle are equal.



Proof

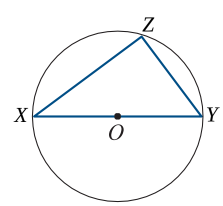
Let $\angle MNP = n$ and $\angle MQP = q$

Then by Theorem 1 $\angle AOB = 2n = 2q$

Therefore $n = q$

Theorem 3

The angle subtended by a diameter at the circumference is equal to a right angle (90°).



Proof

The angle subtended at the centre is 180°.

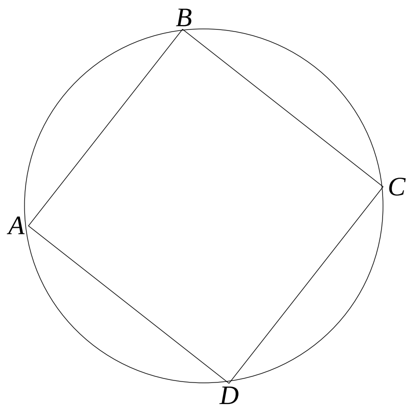
Theorem 1 gives the result.

$$\angle XOY = 180^{o}, \quad \angle XZY = \frac{1}{2}\angle XOY$$

$$\angle XZY = \frac{1}{2}\times 180^\circ = 90^\circ$$

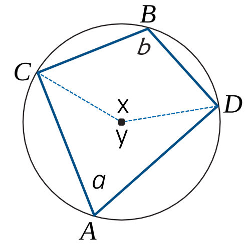
Proved

A quadrilateral which can be inscribed in a circle is called a **cyclic quadrilateral.**



ABCD is a cyclic quadrilateral (four angles shape)

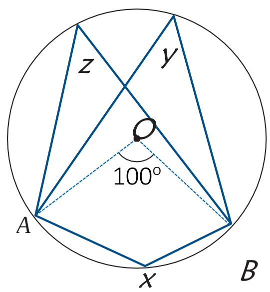
Theorem 4
The opposite angles of a quadrilateral inscribed in a circle sum to two right angles (180). (The opposite angles of a cyclic quadrilateral are supplementary). The converse of this result also holds.



Proof
O is the centre of the circle
By Theorem 1:
$y = 2b$ and $x = 2a$
Also $x + y = 360^\circ$
Therefore $2a + 2b = 360^\circ$
i.e. $a + b = 180^\circ$

The converse states: if a quadrilateral has opposite angles supplementary then the quadrilateral is inscribable in a circle.

Example 1



Find the value of each of the pronumerals in the diagram. O is the centre of the circle and $\angle AOB = 100^\circ$.

Solution

Theorem 1 gives that $z = y = 50^\circ$
The value of x can be found by observing either of the following.

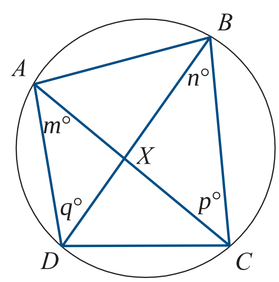
Reflex angle AOB is 260°. Therefore $x = 130^\circ$ (Theorem 1)
or $y + x = 180$ (Theorem 4)
Therefore $x = 180^\circ - 50 = 130^\circ$

Example 2

A, B, C, D are points on a circle. The diagonals of quadrilateral $ABCD$ meet at X. Prove that triangles ADX and BCX are similar.

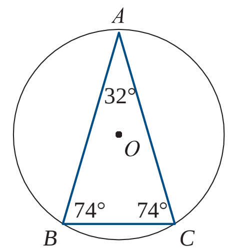
Solution

$\angle DAC$ and $\angle DBC$ are in the same segment.
Therefore $m = n$
$\angle BDA$ and $\angle BCA$ are in the same segment.
Therefore $p = q$
Also $\angle AXD = \angle BXC$ (vertically opposite).
Therefore, triangles ADX and BCX are equiangular and thus similar.



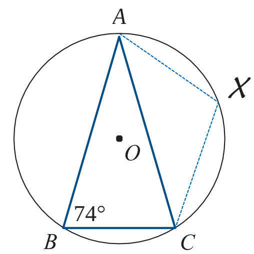
Example 3

An isosceles triangle is inscribed in a circle. Find the angles in the three minor segments of the circle cut off by the sides of this triangle.



Solution

First, to determine the magnitude of $\angle AXC$ cyclic quadrilateral $AXCB$ is formed. Thus $\angle AXC$ and $\angle ABC$ are supplementary.
Therefore $\angle AXC = 106^{\circ}$. All angles in the minor segment formed by AC will have this magnitude.

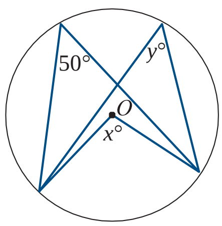


In a similar fashion it can be shown that the angles in the minor segment formed by AB all have magnitude 106^{o} and for the minor segment formed by BC the angles all have magnitude 148°.

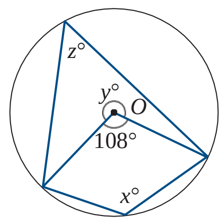
PRACTICE EXERCISE 01

1. Find the values of the pronumerals for each of the following, where O denotes the centre of the given circle.

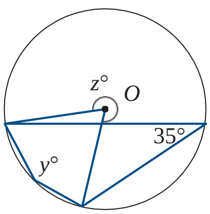
a



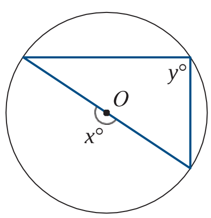
b



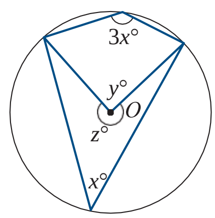
c



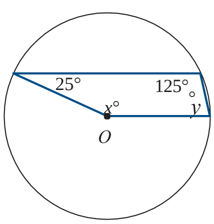
d



e

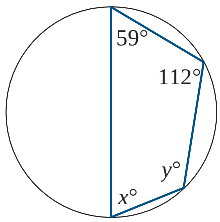


f



2. Find the value of the pronumerals for each of the following.

a



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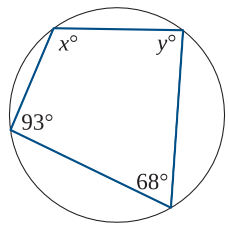

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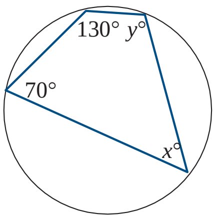
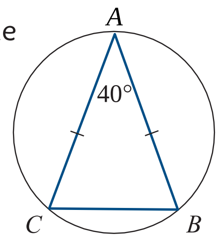


3. An isosceles triangle ABC is inscribed in a circle. What are the angles in the three minor segments cut off by the sides of the triangle?



4. $ABCDE$ is a pentagon inscribed in a circle. If $AE = DE$ and $\angle BDC = 20^\circ$, $\angle CAD = 28^\circ$ and $\angle ABD = 70^\circ$, find all of the interior angles of the pentagon.

5. If two opposite sides of a cyclic quadrilateral are equal, prove that the other two sides are parallel.

6. $ABCD$ is a parallelogram. The circle through A, B and C cuts CD (produced if necessary) at E. Prove that $AE = AD$.

7. $ABCD$ is a cyclic quadrilateral and O is the centre of the circle through A, B, C and D. If $\angle AOC = 120^\circ$, find the magnitude of $\angle ADC$.

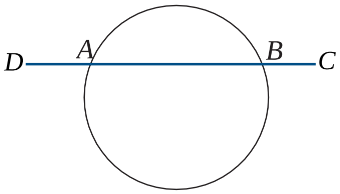
8. Prove that if a parallelogram is inscribed in a circle, it must be a rectangle.

9. Prove that the bisectors of the four interior angles of a quadrilateral form a cyclic quadrilateral.

(b) LINES PROPERTIES OF A CIRCLE

Tangents

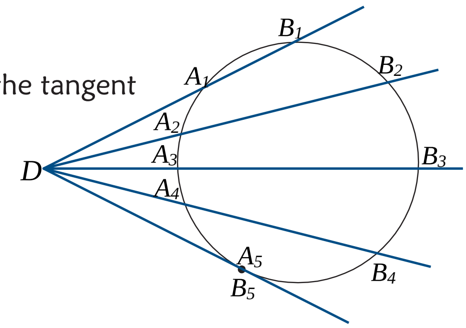
Line DC is called a secant and line segment AB a chord.



If the secant is rotated with D as the pivot point a sequence of pairs of points on the circle is defined. As DC moves towards the edge of the circle the points of the pairs become closer until they eventually coincide.

When PQ is in this final position (i.e., where the intersection points A and B collide) it is called a **tangent** to the circle.

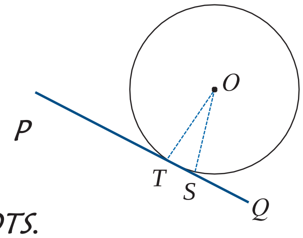
Line D touches the circle. The point at which the tangent touches the circle is called the **point** of



contact. The **length of a tangent** from a point P outside the tangent is the distance between P and the point of contact.

> *Theorem 5*
>
> *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

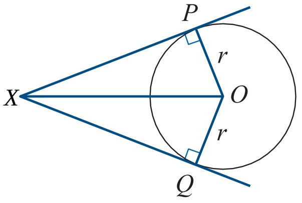
Proof



Let T be the point of contact of tangent PQ.
Let S be the point on PQ, not T,
such that OSP is a right angle.
Triangle OST has a right angle at S.
Therefore $OT > OS$ as OT is the hypotenuse of triangle OTS.
$\therefore$ S is inside the circle as OT is a radius.
$\therefore$ The line through T and S must cut the circle again. But PQ is a tangent. A ***contradiction***.
Therefore $T = S$ and angle OTP is a right angle.

> *Theorem 6*
>
> *The two tangents drawn from an external point to a circle are of the same length.*

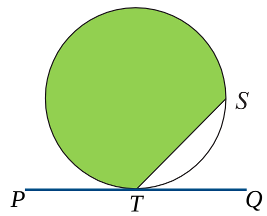
Proof



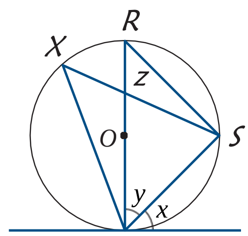
Triangle XPO is congruent to triangle XQO as
XO is a common side. $\angle XPO = \angle XQO = 90^{\circ}$
$OP = OQ$ (radii)
Therefore, $XP = XQ$ (third side of the triangle)

Alternate segment theorem

The shaded segment is called the alternate segment inrelation to $\angle STQ$.
The unshaded segment is alternate to $\angle PTS$



> *Theorem 7*
> *The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.*



Proof

Let $\angle STQ = x^\circ, \angle RTS = y^\circ$ and $\angle TRS = z^\circ$ where RT is a diameter.

Then $\angle RST = 90^\circ$ (Theorem 3, angle subtended by a diameter)

Also $\angle RTQ = 90^\circ$ (Theorem 5, tangent is perpendicular to radius)

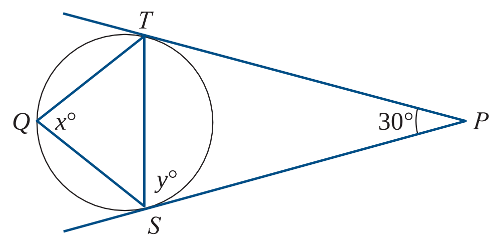
Hence $x + y = 90$ and $y + z = 90$

Therefore $x = z$

But $\angle TXS$ is in the same segment as $\angle TRS$ and therefore $\angle TXS = x$

Example 4

Find the magnitude of the angles x and y in the diagram.



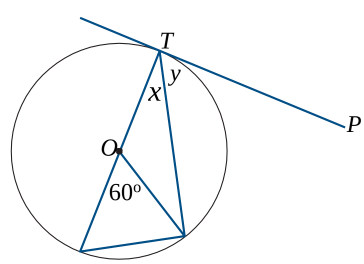
Solution

Triangle PTS is isosceles (Theorem 6, two tangents from the same point) and therefore $\angle PTS = \angle PST$

Hence $y = 75$.
The alternate segment theorem gives that $x = y = 75$

Example 5

Find the values of x and y.
PT is tangent to the circle centre O

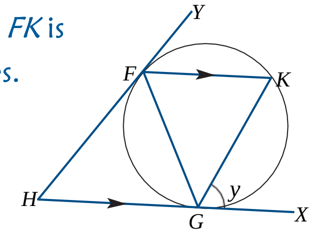


Solution

$x = 30$ as the angle at the circumference is halfthe angle subtended at the centre and $y = 60$ as $\angle OTP$ is a right angle.

Example 6

The tangents to a circle at F and G meet at H. If a chord FK is drawn parallel to HG, prove that triangle FGK is isosceles.



Solution

Let $\angle XGK = y$

Then $\angle GFK = y$ (alternate segment theorem)

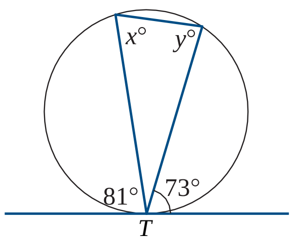
and $\angle GKF = y$ (alternate angles).

Therefore, triangle FGK is isosceles with $FG = KG$

PRACTICE EXERCISE 02

1 Find the value of the pronumerals for each of the following. T is the point of contact of the tangent and O the centre of the circle.

a



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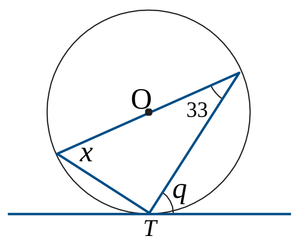
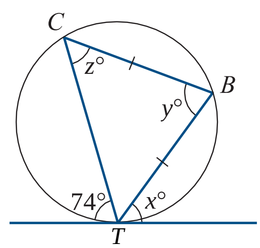
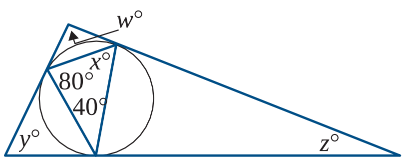


c

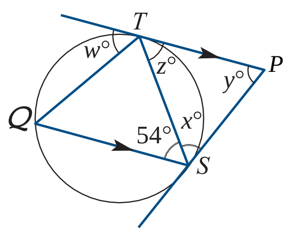


$BC = BT$

d



e



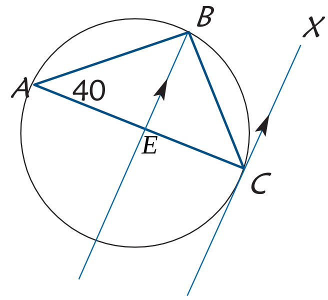
S and T are points of contact of tangents from P. TP is parallel to QS

2 If AB and AC are two tangents to a circle and $\angle BAC = 116^\circ$, find the magnitudes of the angles in the two segments into which BC divides the circle.

3 From a point A outside a circle, a secant ABC is drawn cutting the circle at B and C, and a tangent AD touching it at D. A chord DE is drawn equal in length to chord DB. Prove that triangles ABD and CDE are similar.

4 AB is a chord of a circle and CT, the tangent at C, is parallel to AB. Prove that $CA = CB$.

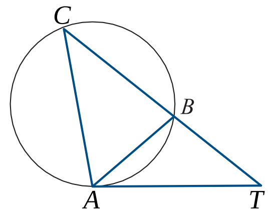
5 A triangle ABC is inscribed in a circle, and the tangent at C to the circle is parallel to the bisector of angle ABC.



a) Find the size of $\angle BCX$.

b) Find the magnitude of $\angle CBE$, where E is the point of intersection of the bisector of angle ABC with AC.

c) Find the magnitude of $\angle ABC$.



6 AT is a tangent at A and TBC is a secant to the circle. Given $\angle CTA = 35^0, \angle CAT = 115^0$, find the magnitude of angles ACB, ABC and BAT.

7 Through a point T, a tangent TA and a secant TPQ are drawn to a circle AQP. If the chord

AB is drawn parallel to PQ, prove that the triangles PAT and BAQ are similar.

8 PQ is a diameter of a circle and AB is a perpendicular chord cutting it at N. Prove that PN isequal in length to the perpendicular from P on to the tangent at A.

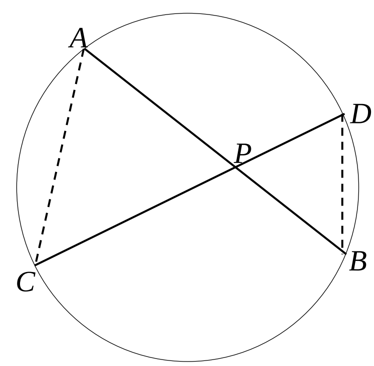
CHORDS IN CIRCLES

Theorem 8

If AB and CD are two chords which cut at a point P (which may be inside or outside the circle)then $PA \times PB = PC \times PD$.

Proof

CASE 1 (The intersection point is inside the circle.)



Consider triangles APC and BPD.

$\angle APC = \angle BPD$ (vertically opposite)

$\angle CDB = \angle CAB$ (angles in the same segment)

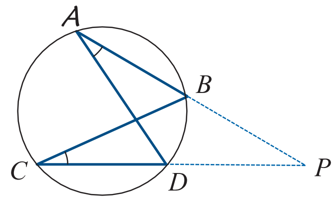
$\angle ACD = \angle DBA$ (angles in the same segment)

Therefore, triangle CAP is similar to triangle BDP.Therefore

$\frac{AP}{PD} = \frac{CP}{PB}$ and $AP \times PB = CP \times PD$,

which can be written $PA \times PB = PC \times PD$

CASE 2 (The intersection point is outside the circle.) Show triangle APD is similar to triangle CPB



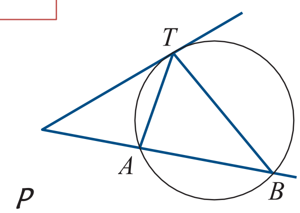
Hence

$$\frac{AP}{CP} = \frac{PD}{PB} \text{ i.e. } AP \times PB = PD \times CP$$

which can be written $PA \times PB = PC \times PD$

Theorem 9

If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB is a secant then $PT^2 = PA \times PB$



Proof

$\angle PTA = \angle TBA$ (alternate segment theorem)

$\angle PTB = \angle TAP$ (angle sum of a triangle)

Therefore, triangle PTB is similar to triangle PAT

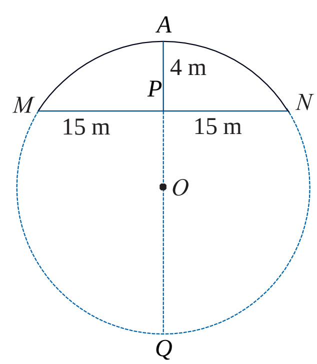
$\therefore \quad \frac{PT}{PA} = \frac{PB}{PT}$ which implies $PT^2 = PA \cdot PB$

Example 7

The arch of a bridge is to be in the form of an arc of a circle. The span of the bridge is to be 30m and the height in the middle 4 m. Find the radius of the circle.

Solution

Theorem 8



$AP \times PQ = MN \times PN$

$4 \times PQ = 15 \times 15$

$PQ = \frac{225}{4} = 56.25$

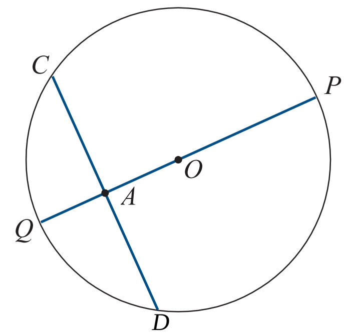
$PQ = 2r - 4 = 56.25$

Therefore, the radius, $r = 30.125$ m

Example 8

If r is the radius of a circle, with center O, and if A is any point inside the circle, show that the product $CA \cdot AD = r^2 - OA^2$, where CD is a chord through A.

Solution



Let PQ be a diameter through A

Theorem 8 gives that

$CA \cdot AD = QA \cdot AP$

Also, $QA = r - OA$ and $PA = r + OA$

$\therefore CA \cdot AD = r^2 - OA^2$

PRACTICE EXERCISE 03

1 If AB is a chord and P is a point on AB such that $AP = 8$ cm, $PB = 5$ cm and P is 3 cm from the centre of the circle, find the radius.

2 If AB is a chord of a circle with centre O and P is a point on AB such that $BP = 4PA$,
$OP = 5$ cm and the radius of the circle is 7 cm, find AB.

3 Two circles intersect at A and B and, from any point P on AB produced tangents PQ and PR are drawn to the circles. Prove that $PQ = PR$.

4 PQ is a variable chord of the smaller of two fixed concentric circles. PQ produced meets the circumference of the larger circle at R. Prove that the product
$RP \times RQ$ *is* constant for all positions and lengths of PQ.

5 Two chords AB and CD intersect at a point P within a circle.

Given that: -

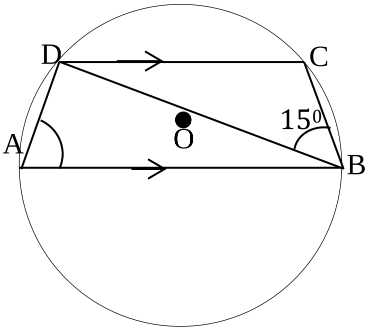
a) $AP = 5$ cm, $PB = 4$ cm, $CP = 2$ cm, find PD

b) $AP = 4$ cm, $CP = 3$ cm, $PD = 8$ cm, find PB.

6 ABC is an isosceles triangle with $AB = AC$. A line through A meets BC at D and the circumcircle of the triangle at E. Prove that $AB^2 = AD \times AE$.

WORKED EXAMPLES

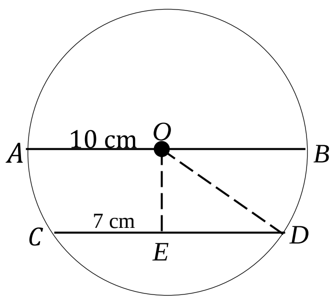
1. If O is the center of the circle, Calculate the size of angle DAB



Solution
Arc DC = 2 × angle DBC
Arc DC = 30°
Arc AD = arc CB (parallel chords subtends equal arcs)
Arc AD + arc DC + arc CB = 180° (semi-circle)
2arc AD + arc DC =180°
2arc AD + 30° = 180°
Arc AD = 75°
Angle DAB is subtended by minor arc DB
Arc DB = arc DC + arc CB
Minor arc DB = 75° + 30° = 105°

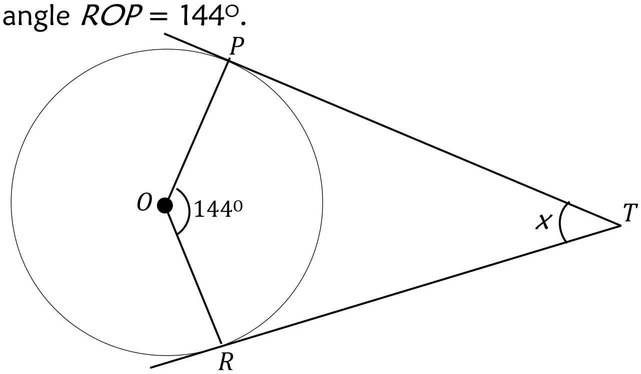
Angle DAB = $\frac{1}{2} \times 105° = 52.5°$

2. A circle of diameter 10 cm has a chord drawn inside it. The chord is 7 cm long.
   a) Make a sketch to show this information.
   b) Calculate the distance from the midpoint of the chord to the centre of the circle.



Construct OD and OE where OD is the radius of the circle and angle OED = 90°, therefore, ΔOED is a right-angled triangle. E bisect the chord CD, thus EC = ED = 3.5cm.
By Pythagoras theorem:
$OD^2 = OE^2 + ED^2$
$5^2 = OE^2 + 3.5^2$
$OE^2 = 12.75$cm, thus OE = 3.6cm

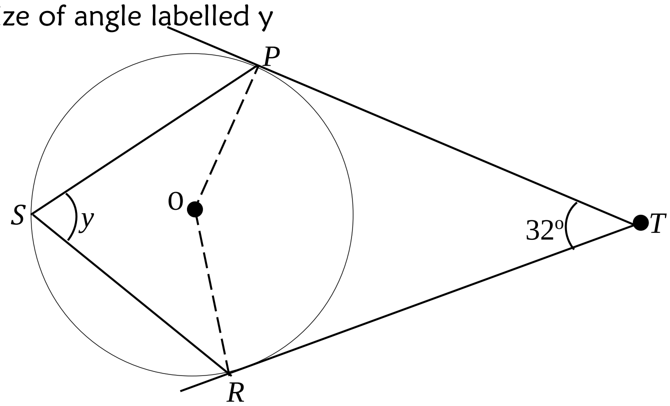
3. The diagram shows the circle with center O. *PT* and *RT* are tangents to the circle, angle *ROP* = 144°.



OP = *OR* (radii)
PT = *TR* (tangents from the external point of a circle)
Angle *OPT* = 90°
Angle *ORT* = 90°
144° + 90° + 90° + x =360° (quadrilateral)
Thus, x = 36°

Work out the size of the angle *PRT* marked x.

4. PT and RT are the tangents to the circle with center O. Angle PTR 32°, find the size of angle labelled y

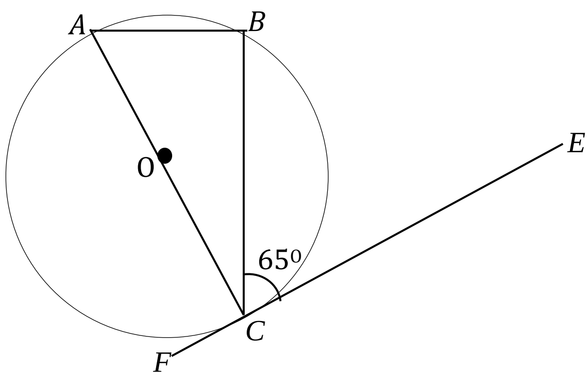


Construct OP and OR
Angles OPT + POR + ORT + RTP = 360°
Angle OPT = ORT = 90° (tangent radius)
180° + 32° POR = 360°
POR = 148°

Angle PSR = $\frac{1}{2}$ angle POR

(theorem)
PSR = y = 74°

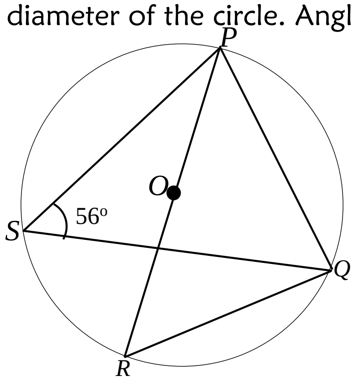
5. Points A, B and C are on the circle and O is the center of the circle. Angle BCE = 63°. FE is a tangent to the circle at point C.



(a) Angle ACB + BCE = 90° (tangent radius)
ACB + 65° = 90°
Angle ACB = 25°
(b) Angle BAC = BCE (alternate segment)
Thus, BAC = 65°

a) Calculate the size of the angle ACB. (Give reasons)
b) Calculate the size of the angle BAC (Give reasons).

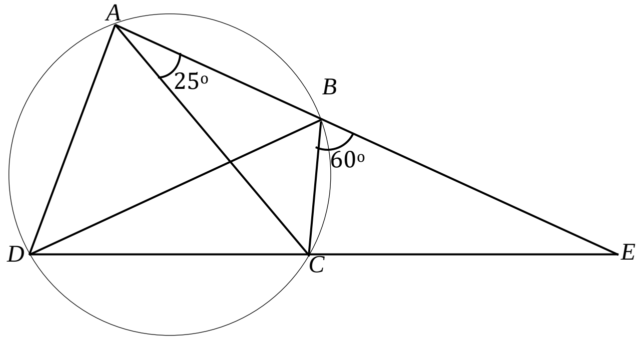
6. P, Q, R and S are points on the circumference of a circle; centre O. PR is a diameter of the circle. Angle PSQ = 56°



(a) PQR = 90° (angle at a semi-circle)
(b) PRQ = 56° (same segment as PSQ)
(c) POQ = 2×56° = 112° (Central-circumference angle)

a) Find the size of angle PQR. (Give a reason for your answer)
b) Find the size of angle PRQ. (Give a reason for your answer)
c) Find the size of angle POQ. (Give a reason for your answer)

7. A, B, C and D are four points on the circumference of a circle.
ABE and DCE are straight lines. Angle BAC=25°. Angle EBC=60° and angle CBD=65°.
a) Find the size of angle ADC.
b) Find the size of angle ADB

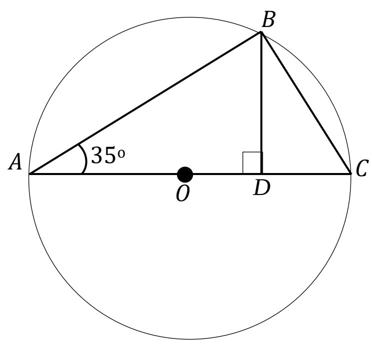


Solution
Angle ABD = 180° – 60° -65° = 55° (straight line)
Angle BCD = 180° – 65° – 25° = 90° (triangle)
Minor arc AB = 2 × 35° = 70°
Minor arc BC = 2 × 25° = 50°
Minor arc AC = Arc AB + Arc BC

(a) ADC = $\frac{1}{2} \times (70° + 50°) = 60°$
(b) Angle ADB = 60° – 25° = 35°
(c) Vanessa is correct, because the angle at semi-circle is 90°, thus angle DCB = 90°

(c) Vanessa says that BD is a diameter of the circle.
Is Vanessa correct? Explain your answer!!

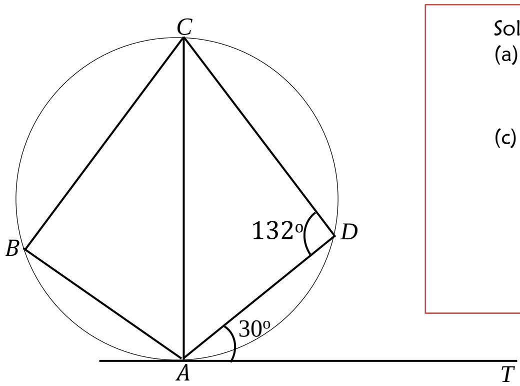
8. The diagram shows a circle, centre O. AC is a diameter. Angle BAC=35°. D is the point on AC such that angle BDA is a right angle.



Solution
(a) ABC = 90° (semi-circle)
35° + 90° + BCA = 180°
thus, angle BCA = 55°
(b) DBC + BCA + CDB = 180°
DBC + 55° + 90° = 180°
DBC = 35°
(c) Angle BOA = 90°

a) Work out the size of angle BCA. Give reasons for your answer.
b) Calculate the size of angle DBC.
c) Calculate the size of angle BOA

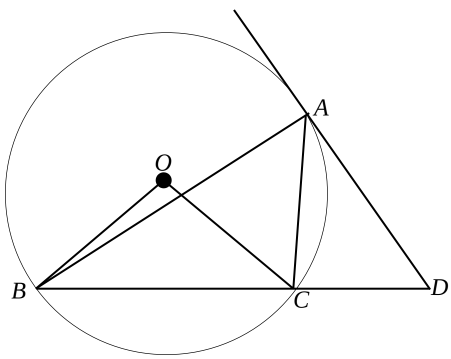
9. A, B, C and D are four points on the circumference of a circle. TA is the tangent to the circle at A. Angle DAT=30°. Angle ADC=132°.



Solution
(a) Angle ABC = 180° – 132° = 48° (opp. Angles of cyclic quadrilateral).
(c) AC cannot be a diameter because
- The semi-circle angle CDA is not 90°, it is 132°
- The other semi-circle angle CBA = 48° so AC cannot be a diameter.

a) Calculate the size of angle ABC. Explain your method.
b) Calculate the size of angle CBD. Explain your method.
c) Explain why AC cannot be a diameter of the circle

10. Points A, B and C lie on the circumference of a circle with centre O. DA is the tangent to the circle at A. BCD is a straight line. OC and AB intersect at E.

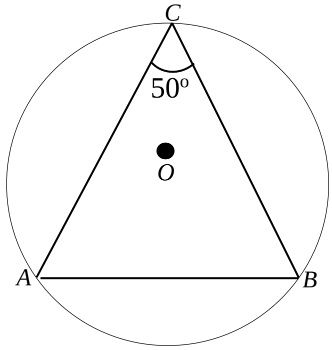


Answers:
(a) BAC = 40°
(b) OBA = 12°

Angle BOC=80°. Angle CAD=38°.

(a) Calculate the size of angle *BAC*.
(b) Calculate the size of angle *OBA*.
(c) Give a reason why it is not possible to draw a circle with diameter *ED* through the point *A*

11. Calculate the area of the minor segment AB if chord AB is 8 cm



Solution
Construct AO and BO (radii)
The central angle AOB
Find the area of the triangle AOB
Find the area of the sector AOB
Area of minor segment = Area of sector – area of triangle
Answer: 10.3 square centimeter

OTHER MATHEMATICS TOPIC NOTES AVAILABLE FOR O-LEVEL

1) All form 1 Mathematics + ICT topics
2) All form 2 mathematics + ICT topics
3) All form 3 mathematics + ICT topics
4) All form 4 mathematics + ICT topics
5) All additional mathematics topics

FOR A-LEVEL

1) All form 5 advance mathematics topics
2) All form 5 BAM topics
3) All form 6 advance mathematics topics
4) All form 6 BAM topics

PHYSICS NOTES AVAILABLE FOR O-LEVEL

1) Physics form 1 full notes
2) Physics form 2 full notes
3) Physics form 3 full notes
4) Physics form 4 full notes

PHYSICS + ICT FOR A-LEVEL

** Preparation in progress until January 2022*

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